BIFURCATION AND STABILITY OF TWO-DIMENSIONAL DOUBLE-DIFFUSIVE CONVECTION

CHUN-HSIUNG HSIA, TIAN MA, AND SHOUHONG WANG

Abstract. In this article, we present a bifurcation and stability analysis on the double-diffusive convection. The main objective is to study 1) the mechanism of the saddle-node bifurcation and hysteresis for the problem, 2) the formation, stability and transitions of the typical convection structures, and 3) the stability of solutions. It is proved in particular that there are two different types of transitions: continuous and jump, which are determined explicitly using some physical relevant nondimensional parameters. It is also proved that the jump transition always leads to the existence of a saddle-node bifurcation and hysteresis phenomena.

1. Introduction

Convective motions occur in a fluid when there are density variations present. Double-diffusive convection is the name given to such convective motions when the density variations are caused by two different components which have different rates of diffusion. Double-diffusion was first originally discovered in the 1857 by Jevons [4], forgotten, and then rediscovered as an "oceanographic curiosity" a century later; see among others Stommel, Arons and Blanchard [12], Veronis [14], and Baines and Gill [1]. In addition to its effects on oceanic circulation, double-diffusion convection has wide applications to such diverse fields as growing crystals, the dynamics of magma chambers and convection in the sun.

The best known double-diffusive instabilities are "salt-fingers" as discussed in the pioneering work by Stern [11]. These arise when hot salty water lies over cold fresh water of a higher density and consist of long fingers of rising and sinking water. A blob of hot salty water which finds itself surrounded by cold fresh water rapidly loses its heat while retaining its salt due to the very different rates of diffusion of heat and salt. The blob becomes cold and salty and hence denser than the surrounding fluid. This tends to make the blob sink further, drawing down more hot salty water from above giving rise to sinking fingers of fluid.

The main objective of this article is to develop a bifurcation and stability theory for the double-diffusive convection, including

1) existence of bifurcations/transitions,
2) asymptotic stability of bifurcated solutions, and
3) the structure/patterns and their stability/transitions in the physical space.

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The analysis is based on a bifurcation theory for nonlinear partial differential equations and a geometric theory of two-dimensional (2D) incompressible flows, both developed recently by two of the authors; see respectively [7, 8] and [9] and the references therein.

This bifurcation theory is centered at a new notion of bifurcation, called attractor bifurcation for dynamical systems, both finite dimensional and infinite dimensional. The main ingredients of the theory include a) the attractor bifurcation theory, b) steady state bifurcation for a class of nonlinear problems with even order non-degenerate nonlinearities, regardless of the multiplicity of the eigenvalues, and c) new strategies for the Lyapunov-Schmidt reduction and the center manifold reduction procedures. The bifurcation theory has been applied to various problems from science and engineering, including, in particular, the Kuramoto-Sivashinshy equation, the Cahn-Hillard equation, the Ginzburg-Landau equation, Reaction-Diffusion equations in Biology and Chemistry, the Bénard convection and the Taylor problem in fluid dynamics.

The geometric theory of 2D incompressible flows was initiated by the authors to study the structure and its stability and transitions of 2-D incompressible fluid flows in the physical spaces. This program of study consists of research in directions: 1) the study of the structure and its transitions/evolutions of divergence-free vector fields, and 2) the study of the structure and its transitions of velocity fields for 2-D incompressible fluid flows governed by the Navier-Stokes equations or the Euler equations. The study in Area 1) is more kinematic in nature, and the results and methods developed can naturally be applied to other problems of mathematical physics involving divergence-free vector fields. In fluid dynamics context, the study in Area 2) involves specific connections between the solutions of the Navier-Stokes or the Euler equations and flow structure in the physical space. In other words, this area of research links the kinematics to the dynamics of fluid flows. This is unquestionably an important and difficult problem. Progresses have been made in several directions. First, a new rigorous characterization of boundary layer separations for 2D viscous incompressible flows is developed recently by the authors, in collaboration in part with Michael Ghil; see [9] and the references therein. Another example in this area is the structure (e.g. rolls) in the physical space in the Rayleigh-Bénard convection, using the structural stability theorem developed in Area 1) together with the application of the aforementioned bifurcation theory; see [6, 9].

In this article, we consider two-dimensional double-diffusive convections modelled by the Boussinesq equations with two diffusion equations of the temperature and salinity functions. In comparison to the Rayleigh-Bénard convection case, and the steady linearized problem around the basic state for the double-diffusive convection is nonsymmetric. This leads to a much harder eigenvalue problem, and consequently much more involved bifurcation and stability analysis. Hence, the bifurcation and the flow structure are much richer.

The central gravity of the analysis is the reduction of the problem to the center manifold in the first unstable eigendirections, based on an approximation formula for the center manifold function. The key idea is to find the approximation of the reduction to certain order, leading to a "nondegenerate" system with higher order perturbations. The full bifurcation and stability analysis are then carried out using

a combination of the attractor bifurcation theory and the geometric theory of 2D incompressible flows.

We now address briefly the main characteristics of the bifurcation and stability analysis of the two-dimensional double-diffusive model presented in this article.

FIRST, the double-diffusive system involves four important nondimensional parameters: the thermal Rayleigh number λ, the solute Rayleigh number η, the Prandtl number σ and the Lewis number τ, defined by (2.5). We examine in this article different transition/instability regimes defined by these parameters. We aim to get a better understanding of the different physical mechanisms involved in the onset of convection. It is hoped that this will enable progress to be made in the theoretical understanding of the onset of double diffusive instabilities.

From the physical point of view, it is natural to consider only the case where the Prandtl number $\sigma > 1$. The Lewis number τ measures the ratio of two diffusivities. From the oceanic circulation point of view, the heat diffuses about 100 times more rapidly than salt [11]; hence $\tau < 1$. In this case, different regimes of stabilities and instabilities/transitions of the basic state can be described by regions in the λ-η plane (the thermal and salt Rayleigh numbers) as shown in Figure 3.1. In this article, we focus on the regimes where

$$\eta < \eta_c = \frac{27}{4}\pi^4\tau^2(1+\sigma^{-1})(1-\tau)^{-1}. \tag{1.1}$$

In the case where $\eta > \eta_c$, transitions to periodic or aperiodic solutions are expected, and will be addressed elsewhere.

SECOND, we show that there are two different transition regimes: continuous and jump, dictated by a nondimensional parameter

$$\eta_{c_1} = \frac{27}{4}\pi^4\tau^3(1-\tau^2)^{-1}. \tag{1.2}$$

For the regime with $\eta < \eta_{c_1}$, the transition is continuous when the thermal Rayleigh number λ crosses a critical value

$$\lambda_c(\eta) = \frac{\eta}{\tau} + \frac{27}{4}\pi^4. \tag{1.3}$$

The rigorous result in this case is stated in Theorem 3.3.

For the regime with $\eta_{c_1} < \eta < \eta_c$, the transition is jump near $\lambda = \lambda_c(\eta)$. It is shown also that for this case, there is a saddle-node bifurcation at $\lambda^*(\eta) < \lambda_c(\eta)$, together with the hysteresis feature of the transition in $\lambda^*(\eta) < \lambda < \lambda_c(\eta)$. The rigorous result in this case is stated in Theorem 3.4, and schematically illustrated by Figure 3.3.

THIRD, as an attractor, the bifurcated attractor has asymptotic stability in the sense that it attracts all solutions with initial data in the phase space outside of the stable manifold of the basic state. As Kirchgässner indicated in [5], an ideal stability theorem would include all physically meaningful perturbations and establish the local stability of a selected class of stationary solutions, and today we are still far from this goal. On the other hand, fluid flows are normally time dependent. Therefore bifurcation analysis for steady state problems provides in general only partial answers to the problem, and is not enough for solving the stability problem. Hence it appears that the right notion of asymptotic stability

should be best described by the attractor near, but excluding, the trivial state. It is one of our main motivations for introducing the attractor bifurcation.

FOURTH, another important aspect of the study is to classify the structure/pattern of the solutions after the bifurcation. A natural tool to attack this problem is the structural stability of the solutions in the physical space. Thanks to the aforementioned geometric theory of 2D incompressible flows, the structure and its transitions of the convection states in the physical space is analyzed, leading in particular to a rigorous justification of the roll structures. More patterns and structures associated with the double diffusive models will be studied elsewhere.

FIFTH, for mathematical completeness and for applications to other physical problems, the bifurcation analysis is also carried out for the case where the Lewis number $\tau > 1$. In this case, only the continuous transition is present. The rigorous result in this case is stated in Theorem 3.5.

This article is organized as follows. The basic governing equations are given in Section 2, and the main theorems are stated in Section 3. The remaining sections are devoted to the proof of the main theorems, with Section 4 on a recapitulation of the attractor bifurcation theory and the geometric theory of incompressible flows, Section 5 on eigenvalue problems, Section 6 on center manifold reductions, and Section 7 on the completion of the proofs.

2. Equations and Set-up

2.1. **Boussinesq equations.** In this paper, we consider the double-diffusive convection problem in a two-dimensional (2D) domain $\mathbb{R}^1 \times (0, h) \subset \mathbb{R}^2$ $(h > 0)$ with coordinates denoted by (x, z). The Boussinesq equations, which govern the motion and states of the fluid flow, are as follows; see Veronis [14]:

$$
\begin{cases}
\dfrac{\partial U}{\partial t} + (U \cdot \nabla)U = -\dfrac{1}{\rho_0}(\nabla p + \rho g e) + \nu \Delta U, \\
\dfrac{\partial T}{\partial t} + (U \cdot \nabla)T = \kappa_T \Delta T, \\
\dfrac{\partial S}{\partial t} + (U \cdot \nabla)S = \kappa_S \Delta S, \\
\operatorname{div} U = 0,
\end{cases}
\tag{2.1}
$$

where $U = (u, w)$ is the velocity function, T is the temperature function, S is the solute concentration, P is the pressure, g is the gravity constant, $e = (0, 1)$ is the unit vector in the z-direction, the constant $\nu > 0$ is the kinematic viscosity, the constant $\kappa_T > 0$ is the thermal diffusity, the constant $\kappa_S > 0$ is the solute diffusity, the constant $\rho_0 > 0$ is the fluid density at the lower surface $z = 0$, and ρ is the fluid density given by the following equation of state

$$
\rho = \rho_0[1 - a(T - T_0) + b(S - S_0)]. \tag{2.2}
$$

Here a and b are assumed to be positive constants. Moreover, the lower boundary $(z = 0)$ is maintained at a constant temperature T_0 and a constant solute concentration S_0, while the upper boundary $(z = h)$ is maintained at a constant temperature T_1 and a constant solute concentration S_1, where $T_0 > T_1$ and $S_0 > S_1$. The trivial

steady state solution of (2.1) - (2.2) is given by

$$\begin{cases} U^0 = 0, \\ T^0 = T_0 - (\dfrac{T_0 - T_1}{h})z, \\ S^0 = S_0 - (\dfrac{S_0 - S_1}{h})z, \\ p^0 = p_0 - g\rho_0[z + \dfrac{a}{2}(\dfrac{T_0 - T_1}{h})z^2 - \dfrac{b}{2}(\dfrac{S_0 - S_1}{h})z^2], \end{cases} \tag{2.3}$$

where p_0 is a constant. To make the equations non-dimensional, we consider the perturbation of the solution from the trivial convection state:

$$U'' = U - U^0, \qquad T'' = T - T^0,$$
$$S'' = S - S^0, \qquad p'' = p - p^0.$$

Then we set

$$(x, z) = h(x', z'), \qquad t = h^2 t'/\kappa_T,$$
$$U'' = \kappa_T U'/h, \qquad T'' = (T_0 - T_1)T',$$
$$S'' = (S_0 - S_1)S', \qquad p'' = \rho_0 \nu \kappa_T p'/h^2.$$

Omitting the primes, the equations (2.1) can be written as

$$\begin{cases} \dfrac{\partial U}{\partial t} = \sigma(\Delta U - \nabla p) + \sigma(\lambda T - \eta S)e - (U \cdot \nabla)U, \\ \dfrac{\partial T}{\partial t} = \Delta T + w - (U \cdot \nabla)T, \\ \dfrac{\partial S}{\partial t} = \tau \Delta S + w - (U \cdot \nabla)S, \\ \text{div} U = 0, \end{cases} \tag{2.4}$$

for (x, z) in the non-dimensional domain $\Omega = \mathbb{R}^1 \times (0, 1)$, where $U = (u, w)$, and the positive nondimensional parameters used above are given by

$$\begin{cases} \lambda = \dfrac{ag(T_0 - T_1)h^3}{\kappa_T \nu} & \text{the thermal Rayleigh number,} \\ \eta = \dfrac{bg(S_0 - S_1)h^3}{\kappa_T \nu} & \text{the salinity Rayleigh number,} \\ \sigma = \dfrac{\nu}{\kappa_T} & \text{the Prandtl number,} \\ \tau = \dfrac{\kappa_S}{\kappa_T} & \text{the Lewis number.} \end{cases} \tag{2.5}$$

We consider periodic boundary condition in the x-direction

$$(U, T, S)(x, z, t) = (U, T, S)(x + 2k\pi/\alpha, z, t) \quad \forall k \in \mathbb{Z}. \tag{2.6}$$

At the top and bottom boundaries, we impose the free-free boundary conditions given by

$$(T, S, w) = 0, \qquad \frac{\partial u}{\partial z} = 0 \quad \text{at} \quad z = 0, 1. \tag{2.7}$$

It's natural to put the constraint

$$\int_\Omega u dx dz = 0, \tag{2.8}$$

for the problem (2.4)-(2.7). It is easy to see from the following computation that (2.4) is invariant under this constraint:

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^1\int_0^{2\pi/\alpha} u dx dz =& \int_0^1\int_0^{2\pi/\alpha} \sigma(u_{xx}+u_{zz}-p_x)-(uu_x+wu_z)dxdz\\ =& \int_0^1 \sigma(u_x-p)\mid_{x=0}^{x=2\pi/\alpha} dz + \int_0^{2\pi/\alpha}\sigma(u_z\mid_{z=0}^{z=1})dx\\ &+\int_0^1\int_0^{2\pi/\alpha}(u_x+w_z)u dx dz\\ =&0.\end{aligned}$$

The initial value conditions are given by

$$(U,T,S)=(\widetilde{U},\widetilde{T},\widetilde{S}) \quad \text{at} \quad t=0. \tag{2.9}$$

2.2. **Functional setting.** Let

$$\begin{aligned} H =&\{(U,T,S)\in L^2(\Omega)^4 \quad \mid \quad \operatorname{div} U=0, w\mid_{z=0,1}=0, \int_\Omega u dx dz=0,\\ &\quad u \text{ is } 2\pi/\alpha\text{-periodic in } x\text{-direction } \},\\ V =&\{(U,T,S)\in H^1(\Omega)^4\cap H \quad \mid \quad (U,T,S) \text{ is } 2\pi/\alpha\text{-periodic in x-direction},\\ &\quad T\mid_{z=0,1}=S\mid_{z=0,1}=0\},\\ H_1 =& V\cap H^2(\Omega).\end{aligned}$$

Let $G: H_1\to H$ and $L_{\lambda\eta}=-A-B_{\lambda\eta}: H_1\to H$ be defined by

$$\begin{aligned} G(\psi) &= (-P[(U\cdot\nabla)U], -(U\cdot\nabla)T, -(U\cdot\nabla)S),\\ A\psi &= (-P[\sigma(\Delta U)], -\Delta T, -\tau\Delta S),\\ B_{\lambda\eta}\psi &= (-P[\sigma(\lambda T-\eta S)e], -w, -w),\end{aligned}$$

for any $\psi=(U,T,S)\in H_1$. Here P is the Leray projection to L^2 fields. Then the Boussinesq equations (2.4)-(2.8) can be written in the following operator form

$$\frac{d\psi}{dt} = L_{\lambda\eta}\psi + G(\psi), \qquad \psi=(U,T,S). \tag{2.10}$$

3. Main Results

3.1. **Definition of attractor bifurcation.** In order to state the main theorems of this article, we proceed with the definition of attractor bifurcation, first introduced by two of the authors in [8, 7].

Let H and H_1 be two Hilbert spaces, and $H_1 \hookrightarrow H$ be a dense and compact inclusion. We consider the following nonlinear evolution equations

$$\begin{cases} \dfrac{du}{dt} = L_\lambda u + G(u,\lambda),\\ u(0)=u_0,\end{cases} \tag{3.1}$$

where $u : [0, \infty) \to H$ is the unknown function, $\lambda \in \mathbb{R}$ is the system parameter, and $L_\lambda : H_1 \to H$ are parameterized linear completely continuous fields depending continuously on $\lambda \in \mathbb{R}^1$, which satisfy

$$
\begin{cases}
-L_\lambda = A + B_\lambda & \text{a sectorial operator,} \\
A : H_1 \to H & \text{a linear homeomorphism,} \\
B_\lambda : H_1 \to H & \text{parameterized linear compact operators.}
\end{cases} \tag{3.2}
$$

It is easy to see [3] that L_λ generates an analytic semi-group $\{e^{tL_\lambda}\}_{t\geq 0}$. Then we can define fractional power operators $(-L_\lambda)^\mu$ for any $0 \leq \mu \leq 1$ with domain $H_\mu = D((-L_\lambda)^\mu)$ such that $H_{\mu_1} \subset H_{\mu_2}$ if $\mu_1 > \mu_2$, and $H_0 = H$.

Furthermore, we assume that the nonlinear terms $G(\cdot, \lambda) : H_\mu \to H$ for some $1 > \mu \geq 0$ are a family of parameterized C^r bounded operators ($r \geq 1$) continuously depending on the parameter $\lambda \in \mathbb{R}^1$, such that

$$
G(u, \lambda) = o(\|u\|_{H_\mu}), \quad \forall\ \lambda \in \mathbb{R}^1. \tag{3.3}
$$

In this paper, we are interested in the sectorial operator $-L_\lambda = A + B_\lambda$ such that there exist an eigenvalue sequence $\{\rho_k\} \subset \mathbb{C}^1$ and an eigenvector sequence $\{e_k, h_k\} \subset H_1$ of A:

$$
\begin{cases}
Az_k = \rho_k z_k, \qquad z_k = e_k + ih_k, \\
\text{Re}\rho_k \to \infty\ (k \to \infty), \\
|\text{Im}\rho_k/(a + \text{Re}\rho_k)| \leq c,
\end{cases} \tag{3.4}
$$

for some $a, c > 0$, such that $\{e_k, h_k\}$ is a basis of H.

Condition (3.4) implies that A is a sectorial operator. For the operator $B_\lambda : H_1 \to H$, we also assume that there is a constant $0 < \theta < 1$ such that

$$
B_\lambda : H_\theta \longrightarrow H \text{ bounded}, \forall\ \lambda \in \mathbb{R}^1. \tag{3.5}
$$

Under conditions (3.4) and (3.5), the operator $-L_\lambda = A + B_\lambda$ is a sectorial operator.

Let $\{S_\lambda(t)\}_{t\geq 0}$ be an operator semi-group generated by the equation (3.1). Then the solution of (3.1) can be expressed as

$$
\psi(t, \psi_0) = S_\lambda(t)\psi_0, \qquad t \geq 0.
$$

Definition 3.1. *A set $\Sigma \subset H$ is called an invariant set of (3.1) if $S(t)\Sigma = \Sigma$ for any $t \geq 0$. An invariant set $\Sigma \subset H$ of (3.1) is said to be an attractor if Σ is compact, and there exists a neighborhood $W \subset H$ of Σ such that for any $\psi_0 \in W$ we have*

$$
\lim_{t\to\infty} \text{dist}_H(\psi(t, \psi_0), \Sigma) = 0.
$$

Definition 3.2. (1) *We say that the solution to equation (3.1) bifurcates from $(\psi, \lambda) = (0, \lambda_0)$ to an invariant set Ω_λ, if there exists a sequence of invariant sets $\{\Omega_{\lambda_n}\}$ of (3.1) such that $0 \notin \Omega_{\lambda_n}$, and*

$$
\lim_{n\to\infty} \lambda_n = \lambda_0,
$$
$$
\lim_{n\to\infty} \max_{x\in\Omega_{\lambda_n}} |x| = 0.
$$

(2) *If the invariant sets Ω_λ are attractors of (3.1), then the bifurcation is called attractor bifurcation.*

3.2. **Main theorems.** We now consider the double diffusive equations (2.4). In this article, we always consider the case where the parameters λ and η satisfying

$$\begin{aligned} &\eta < \eta_c = \frac{27}{4}\pi^4\tau^2(1+\sigma^{-1})(1-\tau)^{-1}, \\ &\lambda \approx \lambda_c = \frac{\eta}{\tau} + \frac{27}{4}\pi^4. \end{aligned} \tag{3.6}$$

First we consider a more physically relevant diffusive regime where the thermal Prandtl number σ is bigger than 1, and the Lewis number τ is less than 1:

$$\sigma > 1 > \tau, \ \alpha^2 = \pi^2/2. \tag{3.7}$$

Here the condition on α defines the aspect ratio of the domain. In this case, we consider two straight lines in the $\lambda - \eta$ parameter plane as shown in Figure 3.1:

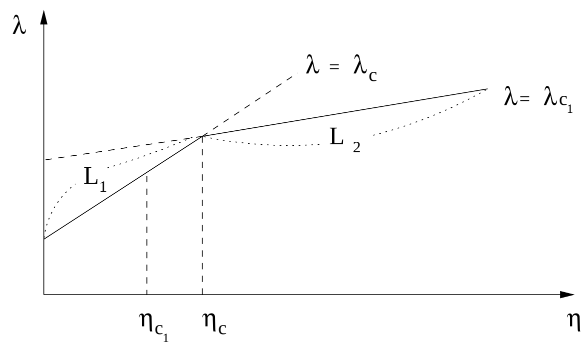

FIGURE 3.1.

$$\begin{cases} L_1: & \lambda = \lambda_c(\eta), \\ L_2: & \lambda = \lambda_{c_1}(\eta), \end{cases} \tag{3.8}$$

where

$$\begin{cases} \lambda_c(\eta) = \dfrac{\eta}{\tau} + \dfrac{27}{4}\pi^4, \\ \lambda_{c_1}(\eta) = \dfrac{(\sigma+\tau)}{(\sigma+1)}\eta + \dfrac{27}{4}\pi^4(1+\sigma^{-1}\tau)(1+\tau), \end{cases} \tag{3.9}$$

Also shown in Figure 3.1 are two critical values for η given by

$$\eta_c = \frac{27}{4}\pi^4\tau^2(1+\sigma^{-1})(1-\tau)^{-1}, \quad \eta_{c_1} = \frac{27}{4}\pi^4\tau^3(1-\tau^2)^{-1}.$$

The following two main theorems study the transitions/bifurcation of the double-diffusive model near the line L_1 for $\eta < \eta_c$.

Theorem 3.3. *Assume that the condition (3.7) holds true, and $\eta < \eta_{c_1} = \frac{27}{4}\pi^4\tau^3(1-\tau^2)^{-1}$. Then the following assertions for the problem (2.4)-(2.8) hold true.*

(1) *If $\lambda \leq \lambda_c$, the steady state $(U,T,S) = 0$ is locally asymptotically stable for the problem.*
(2) *The solutions bifurcate from $((U,T,S),\lambda) = (0,\lambda_c)$ to an attractor Σ_λ for $\lambda > \lambda_c$, which is homeomorphic to S^1, and consists of steady state solutions of the problem.*

(3) *For any* $\psi_0 = (\widetilde{U}, \widetilde{T}, \widetilde{S}) \in H \backslash \Gamma$, *there exists a time* $t_0 \geq 0$ *such that for any* $t \geq t_0$, *the vector field* $U(t, \psi_0)$ *is topologically equivalent to the structure as shown in Figure 3.5 , where* $\psi = (U(t, \psi_0), T(t, \psi_0), S(t, \psi_0))$ *is a solution of (2.4)-(2.8) with (3.7)-(3.6),* Γ *is the stable manifold of the trivial solution* $(U, T, S) = 0$ *with co-dimension 2 in* H.

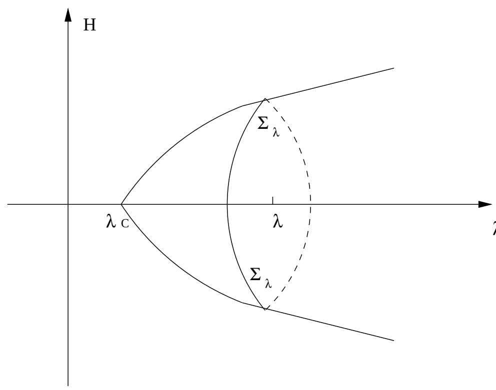

FIGURE 3.2. If $\tau > 1$ or $\eta < \eta_{c_1}$, the equations bifurcate from $(0, \lambda_c)$ to an attractor Σ_λ for $\lambda > \lambda_c$.

Theorem 3.4. *Assume that the condition (3.7) holds true, and* $\eta_c > \eta > \eta_{c_1} = \frac{27}{4}\pi^4\tau^3(1-\tau^2)^{-1}$. *Then there exists a saddle-node bifurcation point* λ_0 $(\lambda_0 < \lambda_c)$ *for the equations, such that the following statements for the problems (2.4)-(2.8) hold true.*

(1) *At* $\lambda = \lambda_0$, *there is an invariant set* $\Sigma_0 = \Sigma_{\lambda_0}$ *with* $0 \notin \Sigma_0$.
(2) *For* $\lambda < \lambda_0$, *there is no invariant set near* Σ_0.
(3) *For* $\lambda_0 < \lambda < \lambda_c$, *there are two branches of invariant sets* Σ^1_λ *and* Σ^2_λ, *and* Σ^2_λ *extends to* $\lambda \geq \lambda_c$ *and near* λ_c *as well.*
(4) *For each* $\lambda > \lambda_0$, Σ^2_λ *is an attractor with* $dist(\Sigma^2_\lambda, 0) > 0$.
(5) *For* $\lambda_0 < \lambda < \lambda_c$,
    (a) Σ^1_λ *is a repeller with* $0 \notin \Sigma^1_\lambda$, *and*
    (b) *when* λ *is near* λ_c, Σ^1_λ *is homeomorphic to* S^1, *consisting of steady states.*

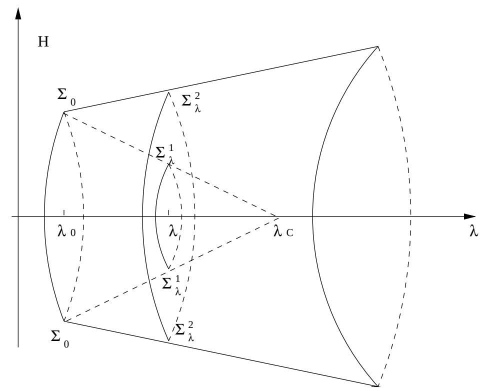

FIGURE 3.3. If $\tau < 1$ and $\eta_{c_1} < \eta < \eta_c$, the equations have a saddle-node bifurcation for $\lambda < \lambda_c$.

We now consider the diffusive parameter regime where $\sigma > 1$, $\tau > 1$, $\alpha^2 = \pi^2/2$ and $\sigma \neq \tau$. In this case, two lines are shown in Figure 3.4. The followin theorem provides bifurcation when λ crosses the line L_1.

Theorem 3.5. *Assume that $\sigma > 1$, $\tau > 1$, $\alpha^2 = \pi^2/2$, $\sigma \neq \tau$ and (3.6) hold, then for any $\eta > 0$, the following assertions for the problem (2.4)-(2.8) hold true.*

(1) *If $\lambda < \lambda_c$, the steady state $(U, T, S) = 0$ is locally asymptotically stable for the problem.*
(2) *The solutions bifurcate from $((U, T, S), \lambda) = (0, \lambda_c)$ to an attractor Σ_λ for $\lambda > \lambda_c$, which is homologically equivalent to S^1, and consists of steady state solutions of the problem.*
(3) *For any $\psi_0 = (\widetilde{U}, \widetilde{T}, \widetilde{S}) \in H \backslash \Gamma$, there exists a time $t_0 \geq 0$ such that for any $t \geq t_0$, the vector field $U(t, \psi_0)$ is topologically equivalent to the structure as shown in Figure 3.5 , where $\psi = (U(t, \psi_0), T(t, \psi_0), S(t, \psi_0))$ is a solution of (2.4)-(2.8), Γ is the stable manifold of the trivial solution $(U, T, S) = 0$ with co-dimension 2 in H.*

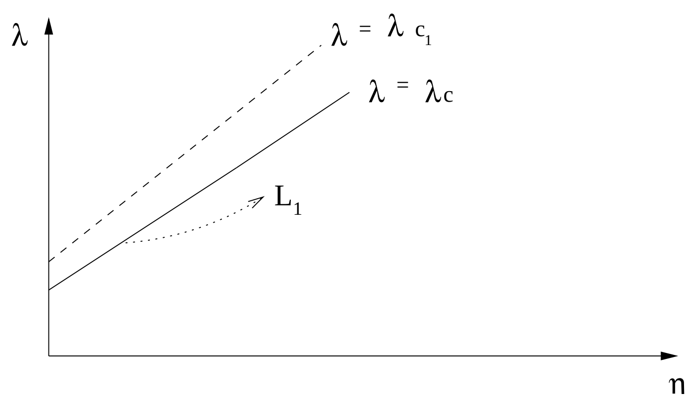

FIGURE 3.4.

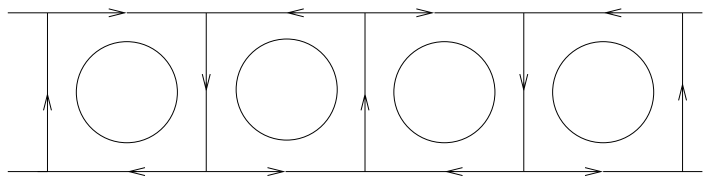

FIGURE 3.5.

4. PRELIMINARIES

4.1. **Attractor bifurcation theory.** Consider (3.1) satisfying (3.2) and (3.3). We start with the Principle of Exchange of Stabilities (PES). Let the eigenvalues (counting the multiplicity) of L_λ be given by

$$\beta_1(\lambda), \beta_2(\lambda), \cdots, \beta_k(\lambda), \cdots \in \mathbb{C}.$$

Suppose that

$$Re\beta_i(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_0 \\ = 0 & \text{if } \lambda = \lambda_0 \\ > 0 & \text{if } \lambda > \lambda_0 \end{cases} \qquad (1 \le i \le m) \tag{4.1}$$

$$Re\beta_j(\lambda_0) < 0. \qquad \forall\, m+1 \le j. \tag{4.2}$$

Let the eigenspace of L_λ at λ_0 be

$$E_0 = \bigcup_{1\le j\le m} \bigcup_{k=1}^{\infty} \{u, v \in H_1 \mid (L_{\lambda_0} - \beta_j(\lambda_0))^k w = 0, w = u + iv\}.$$

It is known that $\dim E_0 = m$.

Theorem 4.1 (T. Ma and S. Wang [8, 7])**.** *Assume that the conditions (3.2)-(3.5) and (4.1)-(4.2) hold true, and $u = 0$ is locally asymptotically stable for (3.1) at $\lambda = \lambda_0$. Then the following assertions hold true.*

(1) *(3.1) bifurcates from $(u, \lambda) = (0, \lambda_0)$ to attractors Σ_λ, having the same homology as S^{m-1}, for $\lambda > \lambda_0$, with $m - 1 \le dim\Sigma_\lambda \le m$, which is connected as $m > 1$;*
(2) *For any $u_\lambda \in \Sigma_\lambda$, u_λ can be expressed as*
$$u_\lambda = v_\lambda + o(\|v_\lambda\|_{H_1}),\ v_\lambda \in E_0;$$
(3) *There is an open set $U \subset H$ with $0 \in U$ such that the attractor Σ_λ bifurcated from $(0, \lambda_0)$ attracts $U\backslash\Gamma$ in H, where Γ is the stable manifold of $u = 0$ with co-dimension m.*

In the case where $m = 2$, the bifurcated attractor can be further classified. Consider a two-dimensional system as follows:

$$\frac{dx}{dt} = \beta(\lambda)x - g(x, \lambda), \quad x \in \mathbb{R}^2. \tag{4.3}$$

Here $\beta(\lambda)$ is a continuous function of λ satisfying

$$\beta(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_0, \\ = 0 & \text{if } \lambda = \lambda_0, \\ > 0 & \text{if } \lambda > \lambda_0, \end{cases} \tag{4.4}$$

and

$$
\begin{cases} g(x,\lambda) = g_k(x,\lambda) + o(|x|^k), \\ g_k(\cdot,\lambda) \text{ is a k-multilinear field}, \\ C_1|x|^{k+1} \leq < g_k(x,\lambda), x > \leq C_2|x|^{k+1}, \end{cases} \tag{4.5}
$$

for some integer $k = 2m+1 \geq 3$, and some constants $C_2 > C_1 > 0$.

The following theorem was proved in [7], which shows that under conditions (4.4) and (4.5), the system (4.3) bifurcates to an S^1-attractor.

Theorem 4.2. *Let the conditions (4.4) and (4.5) hold true. Then the solution to the system (4.3) bifurcates from $(x,\lambda) = (0,\lambda_0)$ to an attractor Σ_λ for $\lambda > \lambda_0$, which is homeomorphic to S^1. Moreover, one and only one of the following is true.*

(1) Σ_λ *is a periodic orbit,*
(2) Σ_λ *consists of only singular points, or*
(3) Σ_λ *contains at most $2(k+1) = 4(m+1)$ singular points, and has $4N+n$ ($N+n \geq 1$) singular points, $2N$ of which are saddle points, $2N$ of which are stable node points (possibly degenerate), and n of which have index zero.*

4.2. **Center manifold reduction.** A crucial ingredient for the proof of the main theorems using the above attractor bifurcation theorems is an approximation formula for center manifold functions derived in [7].

Let H_1 and H be decomposed into

$$
\begin{cases} H_1 = E_1^\lambda \oplus E_2^\lambda, \\ H = \widetilde{E}_1^\lambda \oplus \widetilde{E}_2^\lambda, \end{cases} \tag{4.6}
$$

for λ near $\lambda_0 \in \mathbb{R}^1$, where E_1^λ, E_2^λ are invariant subspaces of L_λ, such that

$$
\begin{aligned} &\dim E_1^\lambda < \infty, \\ &\widetilde{E}_1^\lambda = E_1^\lambda, \\ &\widetilde{E}_2^\lambda = \text{closure of } E_2^\lambda \text{ in } H. \end{aligned}
$$

In addition, L_λ can be decomposed into $L_\lambda = \mathcal{L}_1^\lambda \oplus \mathcal{L}_2^\lambda$ such that for any λ near λ_0,

$$
\begin{cases} \mathcal{L}_1^\lambda = L_\lambda|_{E_1^\lambda} : E_1^\lambda \longrightarrow \widetilde{E}_1^\lambda, \\ \mathcal{L}_2^\lambda = L_\lambda|_{E_2^\lambda} : E_2^\lambda \longrightarrow \widetilde{E}_2^\lambda, \end{cases} \tag{4.7}
$$

where all eigenvalues of $\mathcal{L}_2^\lambda$ possess negative real parts, and the eigenvalues of $\mathcal{L}_1^\lambda$ possess nonnegative real parts at $\lambda = \lambda_0$.

Thus, for λ near λ_0, equation (3.1) can be written as

$$
\begin{cases} \dfrac{dx}{dt} = \mathcal{L}_1^\lambda x + G_1(x,y,\lambda), \\ \dfrac{dy}{dt} = \mathcal{L}_2^\lambda y + G_2(x,y,\lambda), \end{cases} \tag{4.8}
$$

where $u = x + y \in H_1$, $x \in E_1^\lambda$, $y \in E_2^\lambda$, $G_i(x,y,\lambda) = P_i G(u,\lambda)$, and $P_i : H \to \widetilde{E}_i^\lambda$ are canonical projections. Furthermore, let

$$
E_2^\lambda(\mu) = \text{ closure of } E_2^\lambda \text{ in } H_\mu,
$$

with $\mu < 1$ given by (3.3).

By the classical center manifold theorem (see among others [3, 13]), there exists a neighborhood of λ_0 given by $|\lambda-\lambda_0| < \delta$ for some $\delta > 0$, a neighborhood $B_\lambda \subset E_1^\lambda$ of $x = 0$, and a C^1 center manifold function $\Phi(\cdot, \lambda) : B_\lambda \to E_2^\lambda(\theta)$, called the center manifold function, depending continuously on λ. Then to investigate the dynamic bifurcation of (3.1) it suffices to consider the finite dimensional system as follows

$$\frac{dx}{dt} = \mathcal{L}_1^\lambda x + g_1(x, \Phi_\lambda(x), \lambda), \qquad x \in B_\lambda \subset E_1^\lambda. \tag{4.9}$$

Let the nonlinear operator G be in the following form

$$G(u, \lambda) = G_k(u, \lambda) + o(\|u\|^k), \tag{4.10}$$

for some integer $k \geq 2$. Here G_k is a k-multilinear operator

$$\begin{aligned} &G_k : H_1 \times \cdots \times H_1 \longrightarrow H, \\ &G_k(u, \lambda) = G_k(u, \cdots, u, \lambda). \end{aligned}$$

Theorem 4.3. [7] *Under the conditions (4.6), (4.7) and (4.10), the center manifold function* $\Phi(x, \lambda)$ *can be expressed as*

$$\Phi(x, L) = (-\mathcal{L}_2^\lambda)^{-1} P_2 G_k(x, \lambda) + o(\|x\|^k) + O(|\text{Re}\beta|\, \|x\|^k), \tag{4.11}$$

where $\mathcal{L}_2^\lambda$ *is as in (4.7),* $P_2 : H \to \widetilde{E}_2$ *the canonical projection,* $x \in E_1^\lambda$*, and* $\beta = (\beta_1(\lambda), \cdots, \beta_m(\lambda))$ *the eigenvectors of* $\mathcal{L}_1^\lambda$.

Remark 4.4. Suppose that $\{e_j\}_j$, the (generalized) eigenvectors of L_λ, form a basis of H with the dual basis $\{e_j^*\}_j$ such that

$$(e_i, e_j^*)_H \begin{cases} = 0 & \text{if} \quad i \neq j, \\ \neq 0 & \text{if} \quad i = j. \end{cases}$$

Then, we have

$$\begin{aligned} &u = x + y \in E_1^\lambda \oplus E_2^\lambda, \\ &x = \sum_{i=1}^{m} x_i e_i \in E_1^\lambda, \\ &y = \sum_{i=m+1}^{\infty} x_i e_i \in E_2^\lambda. \end{aligned}$$

Hence, near $\lambda = \lambda_0$, $P_2 G_k(x, \lambda)$ can be expressed as follows.

$$P_2 G(x, \lambda) = \sum_{j=m+1}^{\infty} G_k^j(x, \lambda) e_j + o(\|x\|^k), \tag{4.12}$$

where

$$\begin{aligned} &G_k^j(x, \lambda) = \sum_{1 \leq j_1, \cdots, j_k \leq m} a_{j_1 \cdots j_k}^j x_{j_1} \cdots x_{j_k}, \\ &a_{j_1 \cdots j_k}^j = (G_k(e_{j_1}, \cdots, e_{j_k}, \lambda), e_j^*)_H / (e_j, e_j^*). \end{aligned}$$

In many applications, the coefficients $a_{j_1 \cdots j_k}^j$ can be computed, and the first m eigenvalues $\beta_1(\lambda), \cdots, \beta_m(\lambda)$ satisfy

$$|\text{Re}\beta(\lambda_0)| = \sqrt{\sum_{j=1}^{m} (\text{Re}\beta_j(\lambda_0))^2} = 0.$$

Hence (4.11) and (4.12) give an explicit formula for the first approximation of the center manifold functions.

4.3. **Structural stability theorems.** In this subsection, we recall some results on structural stability for 2D divergence-free vector fields developed in [9], which are crucial to study the asymptotic structure in the physical space of the bifurcated solutions of the double-diffusive problem.

Let $C^r(\Omega, \mathbb{R}^2)$ be the space of all C^r $(r \geq 1)$ vector fields on $\Omega = \mathbb{R}^1 \times (0,1)$, which are periodic in x direction with period $2\pi/\alpha$, let $D^r(\Omega, \mathbb{R}^2)$ be the space of all C^r divergence-free vector fields on $\Omega = \mathbb{R}^1 \times (0,1)$, which are periodic in x direction with period $2\pi/\alpha$, and with no normal flow condition in z-direction:

$$D^r(\Omega, \mathbb{R}^2) = \left\{ v = (u, w)(x,z) \in C^r(\Omega, \mathbb{R}^2) \mid w = 0 \text{ at } z = 0, 1 \right\}.$$

Furthermore, we let

$$B_0^r(\Omega, \mathbb{R}^2) = \left\{ v \in D^r(\Omega, \mathbb{R}^2) \mid v = 0 \text{ at } z = 0, 1 \right\},$$

$$B_1^r(\Omega, \mathbb{R}^2) = \left\{ v \in D^r(\Omega, \mathbb{R}^2) \quad \Big| \quad \begin{array}{c} v = 0 \text{ at } z = 0 \\ w = \dfrac{\partial u}{\partial z} = 0 \text{ at } z = 1 \end{array} \right\}.$$

Definition 4.5. *Two vector fields $v_1, v_2 \in C^r(\Omega, \mathbb{R}^2)$ are called topologically equivalent if there exists a homeomorphism of $\varphi : \Omega \to \Omega$, which takes the orbits of v_1 to orbits of v_2 and preserves their orientation.*

Definition 4.6. *Let $X = D^r(\Omega, \mathbb{R}^2)$ or $X = B_0^r(\Omega, \mathbb{R}^2)$. A vector field $v_0 \in X$ is called structurally stable in X if there exists a neighborhood $U \subset X$ of v_0 such that for any $v \in U$, v_0 and v are topologically equivalent.*

Let $v \in D^r(\Omega, \mathbb{R}^2)$. We recall next some basic facts and definitions on divergence–free vector fields.

(1) A point $p \in \Omega$ is called a singular point of v if $v(p) = 0$; a singular point p of v is called non-degenerate if the Jacobian matrix $Dv(p)$ is invertible; v is called regular if all singular points of v are non-degenerate.
(2) An interior non-degenerate singular point of v can be either a center or a saddle, and a non-degenerate boundary singularity must be a saddle.
(3) Saddles of v must be connected to saddles. An interior saddle $p \in \Omega$ is called self-connected if p is connected only to itself, i.e., p occurs in a graph whose topological form is that of the number 8.

Theorem 4.7. *For vector fields satisfying free-free boundary conditions, we set*

$$B_2^r(\Omega, \mathbb{R}^2) = \left\{ v \in D^r(\Omega, \mathbb{R}^2) \;\Big|\; w = \frac{\partial u}{\partial z} = 0 \text{ at } z = 0, 1 \right\},$$

$$B_3^r(\Omega, \mathbb{R}^2) = \left\{ v \in B_2^r(\Omega, \mathbb{R}^2) \quad \Big| \quad \int_\Omega u dx dz = 0 \right\}.$$

Then $v \in B_2^r(\Omega, \mathbb{R}^2)$ (resp.$v \in B_3^r(\Omega, \mathbb{R}^2)$) is structurally stable in $B_2^r(\Omega, \mathbb{R}^2)$ (resp. in $B_2^r(\Omega, \mathbb{R}^2)$) if and only if

1) *v is regular;*
2) *all interior saddle points of v are self-connected; and*
3) *each boundary saddle of v is connected to boundary saddles on the same connected component of $\partial\Omega$ (resp. each boundary saddle of v is connected to boundary saddles not necessarily on the same connected component).*

5. Eigenvalue problem

In order to apply the center manifold theory to reduce the bifurcation problems, we shall analyze the following eigenvalue problem for the linearized equations of (2.4)-(2.8).

$$\begin{cases} \sigma(\Delta U - \nabla p) + (\sigma\lambda T - \sigma\eta S)e = \beta U, \\ \Delta T + w = \beta T, \\ \tau\Delta S + w = \beta S, \\ \mathrm{div} U = 0, \\ \dfrac{\partial u}{\partial z} \mid_{z=0,1} = w \mid_{z=0,1} = T \mid_{z=0,1} = S \mid_{z=0,1} = 0. \end{cases} \tag{5.1}$$

We prove couples of lemmas to show that the operators $-L_{\lambda\mu}$ are sectorial operators when the parameters are properly chosen. In order to get the precise form of the center manifold reduction, the eigenspaces are analyzed in detail in this section.

5.1. **Eigenvalues.** We shall use the method of separation of variables to deal with problem (5.1). Since $\psi = (U, T, S)$ is periodic in x-direction with period $2\pi/\alpha$, we expand the fields in Fourier series as

$$\psi(x, z) = \sum_{j=-\infty}^{\infty} \psi_j(z) e^{ij\alpha x}. \tag{5.2}$$

Plugging (5.2) into (5.1), we obtain the following system of ordinary differential equations

$$\begin{cases} D_j u_j - ij\alpha p_j = \sigma^{-1}\beta u_j, \\ D_j w_j - p_j' + \lambda T_j - \eta S_j = \sigma^{-1}\beta w_j, \\ D_j T_j + w_j = \beta T_j, \\ \tau D_j S_j + w_j = \beta S_j, \\ ij\alpha u_j + w_j' = 0, \\ u_j' \mid_{z=0,1} = w_j \mid_{z=0,1} = T_j \mid_{z=0,1} = S_j \mid_{z=0,1} = 0 \end{cases} \tag{5.3}$$

for $j \in \mathbb{Z}$, where $' = d/dz$, $D_j = d^2/dz^2 - j^2\alpha^2$. We reduce (5.3) to a single equation for $w_j(z)$:

$$\begin{aligned} &\{(\tau D_j - \beta)(D_j - \beta)(D_j - \sigma^{-1}\beta)D_j \\ &\qquad + j^2\alpha^2[\lambda(\tau D_j - \beta) - \eta(D_j - \beta)]\} w_j = 0, \end{aligned} \tag{5.4}$$

$$w_j = w_j'' = w_j^{(4)} = w_j^{(6)} = 0 \quad \text{at} \quad z = 0, 1, \tag{5.5}$$

for $j \in \mathbb{Z}$. Thanks to (5.5), w_j can be expanded in a Fourier sine series

$$w_j(z) = \sum_{l=1}^{\infty} w_{jl} \sin l\pi z \tag{5.6}$$

for $j \in \mathbb{Z}$. Substituting (5.6) into (5.4), we see that the eigenvalues β of the problem (5.1) satisfy the following cubic equations

$$\begin{aligned} &\beta^3 + (\sigma + \tau + 1)\gamma_{jl}^2\beta^2 + [(\sigma + \tau + \sigma\tau)\gamma_{jl}^4 - \sigma j^2\alpha^2\gamma_{jl}^{-2}(\lambda - \eta)]\beta + \sigma\tau\gamma_{jl}^6 \\ &\qquad + \sigma j^2\alpha^2(\eta - \tau\lambda) = 0, \end{aligned} \tag{5.7}$$

for $j \in \mathbb{Z}$ and $l \in \mathbb{N}$, where $\gamma_{jl}^2 = j^2\alpha^2 + l^2\pi^2$.

For the sake of convenience to analyze the distribution of the eigenvalues, we make the following definitions.

Definition 5.1. *For fixed parameters σ, τ, η and λ, let*

(1) $g_{jl}(\beta) = \beta^3 + (\sigma+\tau+1)\gamma_{jl}^2\beta^2 + (\sigma+\tau+\sigma\tau)\gamma_{jl}^4\beta + \sigma\tau\gamma_{jl}^6$,

(2) $h_{jl}(\beta) = [\sigma j^2\alpha^2\gamma_{jl}^{-2}(\lambda-\eta)]\beta - \sigma j^2\alpha^2(\eta-\tau\lambda)$,

(3) $f_{jl}(\beta) = g_{jl}(\beta) - h_{jl}(\beta)$,

(4) $\eta_c = \frac{27}{4}\pi^4\tau^2(1+\sigma^{-1})(1-\tau)^{-1}$,

(5) $\eta_{c_1} = \frac{27}{4}\pi^4\tau^3(1-\tau^2)^{-1}$,

(6) $\lambda_c = \frac{\eta}{\tau} + \frac{27}{4}\pi^4$,

(7) β_{jl1}, β_{jl2} *and* β_{jl3} *be the zeros of* f_{jl} *with* $Re(\beta_{jl1}) \geq Re(\beta_{jl2}) \geq Re(\beta_{jl3})$.

In the following discussions, we shall focus on the following diffusive regime:

$$\sigma > 1 > \tau > 0,\ \alpha^2 = \pi^2/2,\ \eta < \eta_c \text{ and } \lambda \approx \lambda_c. \tag{5.8}$$

Lemma 5.2. (1) *Under the assumption (5.8), $f_{11}(\beta)$ has three simple real zeros.*

(2) *If $\tau > 1$, $\beta_{111} = 0$ is a simple zero of $f_{11}(\beta)$ for $\lambda = \lambda_c$.*

Proof. Since $\lambda \approx \lambda_c$, it suffices to prove this statement for $\lambda = \lambda_c$. In this case,

$$\begin{aligned} f_{11}(\beta) =& \beta^3 + [3\pi^2(\sigma+\tau+1)/2]\beta^2 + [9\pi^4(\sigma+\tau+\sigma\tau)/4 - \sigma(\lambda_c-\eta)/3]\beta \\ =& f(\beta)\beta. \end{aligned}$$

Since $\eta < \eta_c$ or $\tau > 1$, the constant term of $f(\beta)$ is nonzero. Hence $\beta_{111} = 0$ is a simple zero of f_{11}. Moreover, for condition (5.8), the quadratic discriminant of $f(\beta)$ is $9\pi^4(\sigma+1-\tau)^2/4 + 4\sigma\eta(1-\tau)/(3\tau) > 0$. This implies f_{11} has three simple real zeros. $\square$

We summarize the following important lemma about the distribution of the zeros of f_{jl}.

Lemma 5.3. *Assume that either*

1) $\eta < \eta_c$ *with condition (5.8) or*
2) $\eta > 0$ *with* $\tau > 1$,

then

$$\beta_{111}(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_c, \\ = 0 & \text{if } \lambda = \lambda_c, \\ > 0 & \text{if } \lambda > \lambda_c, \end{cases} \tag{5.9}$$

$$Re\beta_{jlk}(\lambda) < 0 \qquad \text{for } (j,l,k) \neq (1,1,1). \tag{5.10}$$

Proof. Let

$$f(\beta) = \beta^3 + a_2\beta^2 + a_1\beta + a_0$$

be a monic real coefficient polynomial of degree 3. The following cases are apparent.

(1) $\beta = 0$ is a zero of $f(\beta)$ if and only if $a_0 = 0$.
(2) $\beta = bi, -bi, a$ $(a, b \in \mathbb{R})$ are zeros of $f(\beta)$ if and only if $a_1 = b^2$, $a_2 = -a$ and $a_0 = -ab^2$.

Case (2) is equivalent to

$$a_1 > 0 \text{ and } a_1 a_2 = a_0.$$

By the above observation, we prove the lemma in several steps as follows.

Step 1. It's easy to see that $\beta = -\gamma_{0l}^2$, $-\sigma\gamma_{0l}^2$, $-\tau\gamma_{0l}^2$ are the zeros of f_{0l}. When $j \neq 0$, $\beta = 0$ is a zero of f_{jl} if and only if

$$\sigma\tau\gamma_{jl}^6 + \sigma j^2\alpha^2(\eta - \tau\lambda) = 0,$$

which is equivalent to

$$\lambda = \frac{\eta}{\tau} + \frac{\gamma_{jl}^6}{j^2\alpha^2}. \tag{5.11}$$

For fixed η, minimizing the right hand side of (5.11), we obtain that $\lambda_c = \frac{\eta}{\tau} + \frac{27}{4}\pi^4$ is the global minimum of λ in Case (1), provided $\alpha^2 = \frac{\pi^2}{2}$ and $(j, l) = (1, 1)$.

Step 2. For Case (2), we obtain in the same fashion as above that

$$\lambda = \frac{(\sigma + \tau)}{(\sigma + 1)}\eta + \frac{\gamma_{jl}^6}{\sigma j^2\alpha^2}(\sigma + \tau)(\tau + 1). \tag{5.12}$$

For a fixed η, minimizing the right hand side of (5.12), we obtain that

$$\lambda_{c_1} = \frac{(\sigma + \tau)}{(\sigma + 1)}\eta + \frac{27}{4}\pi^4(1 + \sigma^{-1}\tau)(1 + \tau)$$

is the global minimum of λ in the case (2), provided $\alpha^2 = \frac{\pi^2}{2}$ and $(j, l) = (1, 1)$.

Step 3. As introduced before, $\lambda = \lambda_c(\eta)$ and $\lambda = \lambda_{c_1}(\eta)$ define two straight lines in the $\lambda - \eta$ plane, shown in Figures 3.1 and 3.4.

If $\tau < 1$, the intersection of the two lines is

$$(\lambda_{c_2}, \eta_c) = (\frac{27}{4}\pi^4\tau(1 + \tau\sigma^{-1})(1 - \tau), \frac{27}{4}\pi^4\tau^2(1 + \sigma^{-1})(1 - \tau)^{-1}).$$

As shown in Figure 3.1, $\lambda_c(\eta) < \lambda_{c_1}(\eta)$ for $\eta < \eta_c$. If $\tau > 1$, $\lambda_c(\eta) < \lambda_{c_1}(\eta)$ for $\eta > 0$; see Figure 3.4. It is easy then to see that in either case, (5.9) and (5.10) hold true. The proof is complete. □

Remark 5.4. (1) This lemma works for the 3D double-diffusive problem as well.
(2) In case 1), if $\eta > \eta_c$ then $\beta_{111} = \bar{\beta}_{112}$ are complex numbers for $\lambda \approx \lambda_{c_1}$.
(3) The distribution of the zeros of f_{jl} was first analyzed by Veronis [14]. The results are scattered in different papers. To make this paper more self-contained, the authors think it's good to summarize it here and give a clear proof.

To check that the operators $-L_{\lambda\eta}$ satisfy condition (3.4), we prove the following lemma.

Lemma 5.5. (1) *Only finitely many numbers of the zeros of $f_{jl}(\beta)$ have nonzero imaginary parts for $(j, l) \in \mathbb{Z} \times \mathbb{N}$.*
(2) $\beta_{jlk} \to -\infty$ *as* $j^2 + l^2 \to \infty$.

Proof. Since $f_{jl} = g_{jl} - h_{jl}$, β is a zero of $f_{jl}(\beta)$ if and only if β satisfies the equation

$$g_{jl}(\beta) = h_{jl}(\beta). \tag{5.13}$$

Plugging $\beta = \gamma_{jl}^2 \beta^*$ into (5.13), we obtain

$$(\beta^* + 1)(\beta^* + \tau)(\beta^* + \sigma) = \vartheta_{jl}[(\lambda - \eta)\beta^* - (\eta - \tau\lambda)], \tag{5.14}$$

where $\vartheta_{jl} = j^2\alpha^2\sigma/\gamma_{jl}^6$. Since $\lim_{j^2+l^2\to\infty} \vartheta_{jl} = 0$, the roots of (5.14) must be negative real numbers near the interval $[-\sigma, -\tau]$ when $(j^2 + l^2)$ is large. This completes the proof. □

5.2. **Eigenvectors.** Let's make some observations to analyze the spectrum. Since $g_{jl}(\beta) = (\beta + \gamma_{jl}^2)(\beta + \tau\gamma_{jl}^2)(\beta + \sigma\gamma_{jl}^2)$ and $h_{jl} = \sigma j^2\alpha^2\gamma_{jl}^{-2}[(\lambda - \eta)\beta - (\eta - \tau\lambda)\gamma_{jl}^2]$, it's easy to check that $\beta = -\gamma_{jl}^2$ or $\beta = -\tau\gamma_{jl}^2$ is a zero of $f_{jl}(\beta)$ if and only if $j = 0$. In the case of $j = 0$, the zeros of f_{jl} are $-\gamma_{jl}^2$, $-\tau\gamma_{jl}^2$ and $-\sigma\gamma_{jl}^2$. The corresponding eigenvectors are

$$\begin{aligned} \psi_{0l}^1(x,z) &= (0, 0, \sin l\pi z, 0)^t, \\ \psi_{0l}^2(x,z) &= (0, 0, 0, \sin l\pi z)^t, \\ \psi_{0l}^3(x,z) &= (\cos l\pi z, 0, 0, 0)^t. \end{aligned} \tag{5.15}$$

To analyze the structures of the eigenspaces of problem (5.1), we make the following definitions.

Definition 5.6. *For $j \neq 0$, we define*

$$\begin{aligned} &\phi_{jl}^1(x,z) = (\frac{l\pi}{j\alpha}\cos j\alpha x \cos l\pi z, \sin j\alpha x \sin l\pi z, 0, 0)^t, \\ &\phi_{jl}^2(x,z) = (0, 0, \sin j\alpha x \sin l\pi z, 0)^t, \;\; \phi_{jl}^3(x,z) = (0, 0, 0, \sin j\alpha x \sin l\pi z)^t, \\ &\phi_{jl}^4(x,z) = (-\frac{l\pi}{j\alpha}\sin j\alpha x \cos l\pi z, \cos j\alpha x \sin l\pi z, 0, 0)^t, \\ &\phi_{jl}^5(x,z) = (0, 0, \cos j\alpha x \sin l\pi z, 0)^t, \;\; \phi_{jl}^6(x,z) = (0, 0, 0, \cos j\alpha x \sin l\pi z)^t. \end{aligned}$$

for each $l \in \mathbb{N}$.

Lemma 5.7. *If $j \neq 0$ and β is a zero of f_{jl}, then we have the followings.*

(1) *The eigenvector corresponding to β in the complexified space is*

$$\psi_{jl}^\beta(x,z) = e^{ij\alpha x}(\frac{il\pi}{j\alpha}\cos l\pi z, \sin l\pi z, A_1(\beta)\sin l\pi z, A_2(\beta)\sin l\pi z)^t, \tag{5.16}$$

$$\text{where} \qquad A_1(\beta) = \frac{1}{\beta + \gamma_{jl}^2} \qquad \text{and} \qquad A_2(\beta) = \frac{1}{\beta + \tau\gamma_{jl}^2}.$$

(2) *If β is a real number, the corresponding eigenvectors are given by*

$$\begin{aligned} \psi_{jl}^{\beta,1} &= \phi_{jl}^1 + A_1(\beta)\phi_{jl}^2 + A_2(\beta)\phi_{jl}^3 \qquad \text{and} \\ \psi_{jl}^{\beta,2} &= \phi_{jl}^4 + A_1(\beta)\phi_{jl}^5 + A_2(\beta)\phi_{jl}^6. \end{aligned} \tag{5.17}$$

(3) *If $Im(\beta) \neq 0$, the generalized eigenvectors corresponding to β and $\bar{\beta}$ are*

$$
\begin{aligned}
\psi_{jl}^{\beta,1} &= \phi_{jl}^1 + R_1(\beta)\phi_{jl}^2 + R_2(\beta)\phi_{jl}^3 + I_1(\beta)\phi_{jl}^5 + I_2(\beta)\phi_{jl}^6, \\
\psi_{jl}^{\beta,2} &= -I_1(\beta)\phi_{jl}^2 - I_2(\beta)\phi_{jl}^3 + \phi_{jl}^4 + R_1(\beta)\phi_{jl}^5 + R_2(\beta)\phi_{jl}^6, \\
\psi_{jl}^{\bar{\beta},1} &= \phi_{jl}^1 + R_1(\bar{\beta})\phi_{jl}^2 + R_2(\bar{\beta})\phi_{jl}^3 + I_1(\bar{\beta})\phi_{jl}^5 + I_2(\bar{\beta})\phi_{jl}^6 \text{ and} \\
\psi_{jl}^{\bar{\beta},2} &= -I_1(\bar{\beta})\phi_{jl}^2 - I_2(\bar{\beta})\phi_{jl}^3 + \phi_{jl}^4 + R_1(\bar{\beta})\phi_{jl}^5 + R_2(\bar{\beta})\phi_{jl}^6,
\end{aligned}
\tag{5.18}
$$

where $R_1(\beta) = Re(A_1(\beta))$, $I_1(\beta) = Im(A_1(\beta))$, $R_2(\beta) = Re(A_2(\beta))$ and $I_2(\beta) = Im(A_2(\beta))$.

The proof of Lemma 5.7 follows from a direct calculation, and we shall omit the details.

Definition 5.8. (1) *If $j = 0$ and $l \in \mathbb{N}$, we define*

$$E_{0l} = span\{\psi_{0l}^1(x,z), \psi_{0l}^2(x,z), \psi_{0l}^3(x,z)\}.$$

(2) *For $j \in \mathbb{N}$, we define $E_{jl}^1 = span\ \{\phi_{jl}^1(x,z), \phi_{jl}^2(x,z), \phi_{jl}^3(x,z)\}$, $E_{jl}^2 = span\ \{\phi_{jl}^4(x,z), \phi_{jl}^5(x,z), \phi_{jl}^6(x,z)\}$ and $E_{jl} = E_{jl}^1 \oplus E_{jl}^2$.*

(3) *For $j \in 0 \cup \mathbb{N}$, $l \in \mathbb{N}$, we define $E_{f_{jl}}$ be the eigenspace spanned by the eigenvectors and the generalized eigenvectors corresponding to the zeros of f_{jl}.*

It is easy to see that the completion of $\oplus_{j=0,l=1}^{\infty} E_{jl}$ in H-norm is H. Hence the following theorem shows that the eigenvectors and the generalized eigenvectors corresponding to the zeros of $\{f_{jl}\}_{j=0,l=1}^{\infty}$ form a basis of H.

Theorem 5.9. *Under the assumption (5.8), we have*

1) *$E_{f_{jl}} = E_{jl}$ for $j \in \{0\} \cup \mathbb{N}$, $l \in \mathbb{N}$; and*
2) *$L_{\lambda\mu}|E_{jl}$ is strictly negative definite for each $(j,l) \in \mathbb{Z} \times \mathbb{N}$ when $\lambda < \lambda_c$.*

Proof. We proceed in two steps.

Step 1. To prove Assertion 1), it is enough to show that dim $E_{f_{jl}} \geq$ dim E_{jl}. The case of $j = 0$ follows from (5.15). When $j \neq 0$, for a fixed l, we examine all the cases as follows.

1. If $\beta_1 > \beta_2 > \beta_3$ are distinct zeros of $f_{jl}(\beta)$, by (5.17), we have dim $E_{f_{jl}} \geq 6 =$ dim E_{jl} .

2. If $\beta_1 = \bar{\beta_2} \in \mathbb{C}\backslash\mathbb{R}$ and β_3 are distinct zeros of $f_{jl}(\beta)$, by (5.17) and (5.18), we have dim $E_{f_{jl}} \geq 6 =$ dim E_{jl} .

3. The first derivative of f_{jl} is $f_{jl}' = 3\beta^2 + 2(\sigma+\tau+1)\gamma_{jl}^2\beta + [(\sigma+\tau+\sigma\tau)\gamma_{jl}^4 - \sigma j^2\alpha^2\gamma_{jl}^{-2}(\lambda-\eta)]$. The quadratic discriminant of f_{jl}' is

$$4\{\frac{1}{2}[(\sigma-\tau)^2 + (\sigma-1)^2 + (\tau-1)^2]\gamma_{jl}^4 + 3\sigma j^2\alpha^2\gamma_{jl}^{-2}(\lambda-\eta)\} > 0,$$

since $\lambda \approx \lambda_c > \eta$. It follows that $f_{jl}(\beta)$ cannot have zeros of multiplicity 3.

4. If $\beta_1 \neq \beta_2 = \beta_3$ are zeros of f_{jl}, direct computation shows

$$
\begin{aligned}
L_{\lambda\eta}\phi_{jl}^2 = & \frac{(\sigma\lambda j^2\alpha^2)}{\gamma_{jl}^2}\frac{(\beta_1+\tau\gamma_{jl}^2)}{(\beta_1-\beta_2)}\psi_{jl}^{\beta_1,1} \\
& + \frac{(\sigma\lambda j^2\alpha^2)}{\gamma_{jl}^2}\frac{(\beta_2+\tau\gamma_{jl}^2)}{(\beta_2-\beta_1)}\psi_{jl}^{\beta_2,1} \\
& + [\frac{(\sigma\lambda j^2\alpha^2)(\tau-1)}{(\beta_1+\gamma_{jl}^2)(\beta_2+\gamma_{jl}^2)} - \gamma_{jl}^2]\phi_{jl}^2
\end{aligned}
\tag{5.19}
$$

Note that

$$
\begin{aligned}
(\beta_1+\gamma_{jl}^2)(\beta_2+\gamma_{jl}^2)(\beta_3+\gamma_{jl}^2) &= -f_{jl}(-\gamma_{jl}^2) \\
&= h_{jl}(-\gamma_{jl}^2) - g_{jl}(-\gamma_{jl}^2) = h_{jl}(-\gamma_{jl}^2) \\
&= \sigma j^2\alpha^2\lambda(\tau-1).
\end{aligned}
$$

Hence

$$\beta_3 = \frac{(\sigma\lambda j^2\alpha^2)(\tau-1)}{(\beta_1+\gamma_{jl}^2)(\beta_2+\gamma_{jl}^2)} - \gamma_{jl}^2.$$

We pick up $v = m\phi_{jl}^2$ to be the generalized eigenvector corresponding to β_3 in E_{jl}^1, where m is some small constant. It's easy to check that $E_{jl}^1 = \text{span}\{\psi_{jl}^{\beta_1,1}, \psi_{jl}^{\beta_2,1}, v\}$ and $L_{\lambda\eta}|E_{jl}^1$ can be represented by matrix

$$
\begin{pmatrix}
\beta_1 & 0 & m\frac{(\sigma\lambda j^2\alpha^2)}{\gamma_{jl}^2}\frac{(\beta_1+\tau\gamma_{jl}^2)}{(\beta_1-\beta_2)} \\
0 & \beta_2 & m\frac{(\sigma\lambda j^2\alpha^2)}{\gamma_{jl}^2}\frac{(\beta_2+\tau\gamma_{jl}^2)}{(\beta_2-\beta_1)} \\
0 & 0 & \beta_3
\end{pmatrix}
\tag{5.20}
$$

in the basis $\{\psi_{jl}^{\beta_1,1}, \psi_{jl}^{\beta_2,1}, v\}$. The same argument works for E_{jl}^2 as well.

Step 2. It's easy to check that E_{jl}^1 is orthogonal to E_{jl}^2 for $(j,l) \in \mathbb{N}\times\mathbb{N}$ and $E_{j_1l_1}$ is orthogonal to $E_{j_2l_2}$ for $(j_1,l_1) \neq (j_2,l_2)$. Lemma 5.3 together with Step1 imply that $L_{\lambda\eta}|E_{jl}$ is strictly negative definite when $\lambda < \lambda_c$. This completes the proof. □

Lemma 5.3 and Lemma 5.5 together with Theorem 5.9 imply the following theorem.

Theorem 5.10. *Under assumption (5.8), $-L_{\lambda\eta}$ is a sectorial operator.*

Remark 5.11.
(1) Since dim $E_{jl} = 3$ or 6 which is finite, there exists a vector $\Psi_{jl}^{\beta,k} \in E_{jl}$ such that

$$< \Psi_{jl}^{\beta,k}, \psi_{jl}^{\beta^*,k^*} >_H \begin{cases} = 0 & \text{for } (\beta,k) \neq (\beta^*,k^*), \\ \neq 0 & \text{for } (\beta,k) = (\beta^*,k^*), \end{cases}$$

where β and β^* are zeros of f_{jl} and $k, k^* = 1, 2$.
(2) Note that E_{jl}^1 is orthogonal to E_{jl}^2 for $(j,l) \in \mathbb{N}\times\mathbb{N}$ and $E_{j_1l_1}$ is orthogonal to $E_{j_2l_2}$ for $(j_1,l_1) \neq (j_2,l_2)$, hence we conclude that $< \Psi_{j_1l_1}^{\beta,k_1}, \psi_{j_2l_2}^{\beta^*,k_2} >_H = 0$ for $(j_1,l_1,\beta,k_1) \neq (j_2,l_2,\beta^*,k_2)$.

(3) If $j=0$, we pick up $\Psi^k_{0l}=\psi^k_{0l}$. For $(j,l)=(1,1)$, we pick up

(5.21) $$\Psi_{11}^{\beta_{111},1}=\phi^1_{11}+C_1\phi^2_{11}+C_2\phi^3_{11}, \qquad \text{and} \qquad \Psi_{11}^{\beta_{111},2}=\phi^4_{11}+C_1\phi^5_{11}+C_2\phi^6_{11},$$

where

(5.22) $$C_1=\frac{\sigma\lambda}{\beta_{111}+\gamma_{11}^2} \qquad \text{and} \qquad C_2=\frac{-\sigma\eta}{\beta_{111}+\tau\gamma_{11}^2}.$$

(4) Lemma 5.2 and Theorem 5.9 show that the multiplicity of the eigenvalue $\beta_{111}(\lambda)$ is two and the corresponding eigenvectors are $\psi_{11}^{\beta_{111},1}$ and $\psi_{11}^{\beta_{111},2}$.

(5) For $\psi\in H_{3/4}\subset H$, by Sobolev inequality,

$$|G(\psi)|^2_H\leq\int_0^1\int_0^{2\pi/\alpha}|\psi|^2|\nabla\psi|^2dxdz\leq|\psi|^2_{L^\infty}|\psi|^2_{H_{1/2}}\leq C|\psi|^4_{H_{3/4}}.$$

where C is some constant. Hence, $G(\psi)=o(|\psi|_{H_{3/4}})$.

6. Center manifold reduction

We are now in a position to reduce equations of (2.4)- (2.8) to the center manifold. We would like to fix $\eta<\eta_c$, and let $\lambda\approx\lambda_c$ be the bifurcation parameter. For any $\psi=(U,T,S)\in H$, we have

$$\psi=\sum_{j=0,l=1}^{\infty}\sum_{k=1}^{3}(x_{jlk}\psi_{jl}^{\beta_{jlk},1}+y_{jlk}\psi_{jl}^{\beta_{jlk},2}).$$

Since β_{111} is the first eigenvalue, $\psi_{11}^{\beta_{111},1}$ and $\psi_{11}^{\beta_{111},2}$ are the first eigenvectors. The reduced equations are given by

(6.1) $$\begin{cases}\dfrac{dx_{111}}{dt}=\beta_{111}(\lambda)x_{111}+\dfrac{1}{<\psi_{11}^{\beta_{111},1},\Psi_{11}^{\beta_{111},1}>_H}<G(\psi,\psi),\Psi_{11}^{\beta_{111},1}>_H,\\[2mm] \dfrac{dy_{111}}{dt}=\beta_{111}(\lambda)y_{111}+\dfrac{1}{<\psi_{11}^{\beta_{111},2},\Psi_{11}^{\beta_{111},2}>_H}<G(\psi,\psi),\Psi_{11}^{\beta_{111},2}>_H\ .\end{cases}$$

Here for $\psi_1=(U_1,T_1,S_1)$, $\psi_2=(U_2,T_2,S_2)$ and $\psi_3=(U_3,T_3,S_3)$,

$$G(\psi_1,\psi_2)=-(P(U_1\cdot\nabla)U_2,(U_1\cdot\nabla)T_2,(U_1\cdot\nabla)S_2)^t \qquad \text{and}$$

$$<G(\psi_1,\psi_2),\psi_3>_H=-\int_\Omega[<(U_1\cdot\nabla)U_2,U_3>_{\mathbb{R}^2}+(U_1\cdot\nabla)T_2T_3+(U_1\cdot\nabla)S_2S_3]dxdz,$$

where P is the Leray projection to L^2 fields.

Let the center manifold function be denoted by

(6.2) $$\Phi=\sum_{\beta\neq\beta_{111}}(\Phi_{jl}^{\beta,1}(x_{111},y_{111})\psi_{jl}^{\beta,1}+\Phi_{jl}^{\beta,2}(x_{111},y_{111})\psi_{jl}^{\beta,2}).$$

Note that for any $\psi_i\in H_1(i=1,\,2,\,3)$,

$$<G(\psi_1,\psi_2),\psi_2>_H=0,$$
$$<G(\psi_1,\psi_2),\psi_3>_H=-<G(\psi_1,\psi_3),\psi_2>_H;$$

and for k=1,2,

$$<G(\psi_1,\psi_{11}^{\beta_{111},k}),\Psi_{11}^{\beta_{111},k}>_H=0.$$

Then by $\psi=x_{111}\psi_{11}^{\beta_{111},1}+y_{111}\psi_{11}^{\beta_{111},2}+\Phi$, we have

$$(6.3)\qquad \begin{aligned} < G(\psi,\psi), \Psi_{11}^{\beta_{111},1} >_H = & < G(\psi_{11}^{\beta_{111},2}, \psi_{11}^{\beta_{111},2}), \Psi_{11}^{\beta_{111},1} >_H y_{111}^2 \\ & + < G(\psi_{11}^{\beta_{111},1}, \psi_{11}^{\beta_{111},2}), \Psi_{11}^{\beta_{111},1} >_H x_{111}y_{111} \\ & - < G(\psi_{11}^{\beta_{111},1}, \Psi_{11}^{\beta_{111},1}), \Phi >_H x_{111} \\ & - < G(\psi_{11}^{\beta_{111},2}, \Psi_{11}^{\beta_{111},1}), \Phi >_H y_{111} \\ & + < G(\Phi, \psi_{11}^{\beta_{111},2}), \Psi_{11}^{\beta_{111},1} >_H y_{111} \\ & + < G(\Phi, \Phi), \Psi_{11}^{\beta_{111},1} >_H, \end{aligned}$$

$$(6.4)\qquad \begin{aligned} < G(\psi,\psi), \Psi_{11}^{\beta_{111},2} >_H = & < G(\psi_{11}^{\beta_{111},1}, \psi_{11}^{\beta_{111},1}), \Psi_{11}^{\beta_{111},2} >_H x_{111}^2 \\ & + < G(\psi_{11}^{\beta_{111},2}, \psi_{11}^{\beta_{111},1}), \Psi_{11}^{\beta_{111},2} >_H x_{111}y_{111} \\ & - < G(\psi_{11}^{\beta_{111},2}, \Psi_{11}^{\beta_{111},2}), \Phi >_H y_{111} \\ & - < G(\psi_{11}^{\beta_{111},1}, \Psi_{11}^{\beta_{111},2}), \Phi >_H x_{111} \\ & + < G(\Phi, \psi_{11}^{\beta_{111},1}), \Psi_{11}^{\beta_{111},2} >_H x_{111} \\ & + < G(\Phi, \Phi), \Psi_{11}^{\beta_{111},1} >_H, \end{aligned}$$

By direct calculations, we obtain that

(6.5)

$$\begin{aligned} G(\psi_{11}^{\beta_{111},1}, \psi_{11}^{\beta_{111},1}) &= -(P(-\frac{\pi^2}{2\alpha}\sin 2\alpha x, \frac{\pi}{2}\sin 2\pi z), \frac{A_1(\beta_{111})\pi}{2}\sin 2\pi z, \frac{A_2(\beta_{111})\pi}{2}\sin 2\pi z)^t, \\ G(\psi_{11}^{\beta_{111},1}, \psi_{11}^{\beta_{111},2}) &= -(P(\frac{-\pi^2}{2\alpha}(\cos 2\alpha x + \cos 2\pi z), 0), 0, 0)^t, \\ G(\psi_{11}^{\beta_{111},2}, \psi_{11}^{\beta_{111},1}) &= -(P(\frac{-\pi^2}{2\alpha}(\cos 2\alpha x - \cos 2\pi z), 0), 0, 0)^t, \\ G(\psi_{11}^{\beta_{111},2}, \psi_{11}^{\beta_{111},2}) &= -(P(\frac{\pi^2}{2\alpha}\sin 2\alpha x, \frac{\pi}{2}\sin 2\pi z), \frac{A_1(\beta_{111})\pi}{2}\sin 2\pi z, \frac{A_2(\beta_{111})\pi}{2}\sin 2\pi z)^t, \end{aligned}$$

and

(6.6)

$$\begin{aligned} G(\psi_{11}^{\beta_{111},1}, \Psi_{11}^{\beta_{111},1}) &= -(P(-\frac{\pi^2}{2\alpha}\sin 2\alpha x, \frac{\pi}{2}\sin 2\pi z), \frac{C_1(\beta_{111})\pi}{2}\sin 2\pi z, \frac{C_2(\beta_{111})\pi}{2}\sin 2\pi z)^t, \\ G(\psi_{11}^{\beta_{111},1}, \Psi_{11}^{\beta_{111},2}) &= -(P(\frac{-\pi^2}{2\alpha}(\cos 2\alpha x + \cos 2\pi z), 0), 0, 0)^t, \\ G(\psi_{11}^{\beta_{111},2}, \Psi_{11}^{\beta_{111},1}) &= -(P(\frac{-\pi^2}{2\alpha}(\cos 2\alpha x - \cos 2\pi z), 0), 0, 0)^t, \\ G(\psi_{11}^{\beta_{111},2}, \Psi_{11}^{\beta_{111},2}) &= -(P(\frac{\pi^2}{2\alpha}\sin 2\alpha x, \frac{\pi}{2}\sin 2\pi z), \frac{C_1(\beta_{111})\pi}{2}\sin 2\pi z, \frac{C_2(\beta_{111})\pi}{2}\sin 2\pi z)^t. \end{aligned}$$

By (6.5) and (6.6), we derive that for $(j,l) \neq (0,2)$,

$$(6.7)\qquad \begin{aligned} &< G(\psi_{11}^{\beta_{111},k_1}, \psi_{11}^{\beta_{111},k_2}), \Psi_{jl}^{\beta,k} >_H = 0, \\ &< G(\psi_{jl}^{\beta,k}, \psi_{11}^{\beta_{111},k_1}), \Psi_{11}^{\beta_{111},k_2} >_H = 0, \end{aligned}$$

where $k_1, k_2 = 1, 2$. Hence the first two terms in (6.3) and (6.4) are gone. Since the center manifold function contains only higher order terms

$$\Phi(x_{111}, y_{111}) = O(|x_{111}|^2, |y_{111}|^2),$$

we derive that

$$\begin{cases} < G(\Phi, \Phi), \Psi_{11}^{\beta_{111},1} >_H = o(|x_{111}|^3, |y_{111}|^3), \\ < G(\Phi, \Phi), \Psi_{11}^{\beta_{111},2} >_H = o(|x_{111}|^3, |y_{111}|^3). \end{cases} \tag{6.8}$$

By (6.7) and (6.8), only $\Phi_{02}^{\beta_1}(x_{111}, y_{111})$, $\Phi_{02}^{\beta_2}(x_{111}, y_{111})$ and $\Phi_{02}^{\beta_3}(x_{111}, y_{111})$ (where $\beta_1 = -\gamma_{02}^2 = -4\pi^2$, $\beta_2 = -\tau\gamma_{02}^2 = -4\tau\pi^2$, and $\beta_3 = -\sigma\gamma_{02}^2 = -4\sigma\pi^2$.) contribute to the third order terms in evaluation of (6.3) and (6.4). Direct calculations show

$$\begin{aligned}
&< G(\psi_{11}^{\beta_{111},1}, \psi_{11}^{\beta_{111},1}), \Psi_{02}^{\beta_1} >_H = \int_0^1 \int_0^{\frac{2\pi}{\alpha}} -\frac{A_1(\beta_{111})\pi}{2} \sin^2 2\pi z dx dz = \frac{-A_1(\beta_{111})\pi^2}{2\alpha}, \\
&< G(\psi_{11}^{\beta_{111},2}, \psi_{11}^{\beta_{111},2}), \Psi_{02}^{\beta_1} >_H = \int_0^1 \int_0^{\frac{2\pi}{\alpha}} -\frac{A_1(\beta_{111})\pi}{2} \sin^2 2\pi z dx dz = \frac{-A_1(\beta_{111})\pi^2}{2\alpha}, \\
&< G(\psi_{11}^{\beta_{111},2}, \psi_{11}^{\beta_{111},1}), \Psi_{02}^{\beta_1} >_H = 0, \qquad < G(\psi_{11}^{\beta_{111},1}, \psi_{11}^{\beta_{111},2}), \Psi_{02}^{\beta_1} >_H = 0, \\
&< G(\psi_{11}^{\beta_{111},1}, \psi_{11}^{\beta_{111},1}), \Psi_{02}^{\beta_2} >_H = \int_0^1 \int_0^{\frac{2\pi}{\alpha}} -\frac{A_2(\beta_{111})\pi}{2} \sin^2 2\pi z dx dz = \frac{-A_2(\beta_{111})\pi^2}{2\alpha}, \\
&< G(\psi_{11}^{\beta_{111},2}, \psi_{11}^{\beta_{111},2}), \Psi_{02}^{\beta_2} >_H = \int_0^1 \int_0^{\frac{2\pi}{\alpha}} -\frac{A_2(\beta_{111})\pi}{2} \sin^2 2\pi z dx dz = \frac{-A_2(\beta_{111})\pi^2}{2\alpha}, \\
&< G(\psi_{11}^{\beta_{111},2}, \psi_{11}^{\beta_{111},1}), \Psi_{02}^{\beta_2} >_H = 0, \qquad < G(\psi_{11}^{\beta_{111},1}, \psi_{11}^{\beta_{111},2}), \Psi_{02}^{\beta_2} >_H = 0, \\
&< G(\psi_{11}^{\beta_{111},1}, \psi_{11}^{\beta_{111},1}), \Psi_{02}^{\beta_3} >_H = 0, \qquad < G(\psi_{11}^{\beta_{111},2}, \psi_{11}^{\beta_{111},2}), \Psi_{02}^{\beta_3} >_H = 0, \\
&< G(\psi_{11}^{\beta_{111},2}, \psi_{11}^{\beta_{111},1}), \Psi_{02}^{\beta_3} >_H = \frac{\pi^3}{2\alpha^2}, \qquad < G(\psi_{11}^{\beta_{111},1}, \psi_{11}^{\beta_{111},2}), \Psi_{02}^{\beta_3} >_H = \frac{-\pi^3}{2\alpha^2}, \\
&< \psi_{02}^{\beta_1}, \Psi_{02}^{\beta_1} >_H = < \psi_{02}^{\beta_2}, \Psi_{02}^{\beta_2} >_H = < \psi_{02}^{\beta_3}, \Psi_{02}^{\beta_3} >_H = \frac{\pi}{\alpha}.
\end{aligned}$$

Applying Theorem 4.3, we obtain

$$\begin{aligned}
&\Phi_{02}^{\beta_1}(x_{111}, y_{111}) = \frac{A_1(\beta_{111})\pi}{2\beta_1}(x_{111}^2 + y_{111}^2) + o(x_{111}^2 + y_{111}^2), \\
&\Phi_{02}^{\beta_2}(x_{111}, y_{111}) = \frac{A_2(\beta_{111})\pi}{2\beta_2}(x_{111}^2 + y_{111}^2) + o(x_{111}^2 + y_{111}^2), \\
&\Phi_{02}^{\beta_3}(x_{111}, y_{111}) = o(x_{111}^2 + y_{111}^2).
\end{aligned} \tag{6.9}$$

By (6.5)-(6.9), we evaluate

$$
(6.10)\qquad \begin{cases}
< G(\psi_{11}^{\beta_{111},1}, \Psi_{11}^{\beta_{111},1}), \Phi >_H \\
=< G(\psi_{11}^{\beta_{111},1}, \Psi_{11}^{\beta_{111},1}), \Phi_{02}^{\beta_1}\psi_{02}^{\beta_1} + \Phi_{02}^{\beta_2}\psi_{02}^{\beta_2} >_H + o(x_{111}^2 + y_{111}^2) \\
= \dfrac{-C_1\pi^2}{2\alpha}\Phi_{02}^{\beta_1} - \dfrac{C_2\pi^2}{2\alpha}\Phi_{02}^{\beta_2} + o(x_{111}^2 + y_{111}^2) \\
= -\dfrac{\pi^3}{4\alpha}(\dfrac{A_1C_1}{\beta_1} + \dfrac{A_2C_2}{\beta_2})(x_{111}^2 + y_{111}^2) + o(x_{111}^2 + y_{111}^2) \\
= \dfrac{1}{8\sqrt{2}}(A_1C_1 + A_2C_2\tau^{-1})(x_{111}^2 + y_{111}^2) + o(x_{111}^2 + y_{111}^2), \\
< G(\psi_{11}^{\beta_{111},2}, \Psi_{11}^{\beta_{111},1}), \Phi >_H = o(x_{111}^2 + y_{111}^2), \\
< G(\Phi, \psi_{11}^{\beta_{111},2}), \Psi_{11}^{\beta_{111},1} >_H = o(x_{111}^2 + y_{111}^2), \\
< \psi_{11}^{\beta_{111},1}, \Psi_{11}^{\beta_{111},1} >_H = \dfrac{1}{\sqrt{2}}(3 + A_1C_1 + A_2C_2),
\end{cases}
$$

$$
(6.11)\qquad \begin{cases}
< G(\psi_{11}^{\beta_{111},2}, \Psi_{11}^{\beta_{111},2}), \Phi >_H \\
=< G(\psi_{11}^{\beta_{111},1}, \Psi_{11}^{\beta_{111},2}), \Phi_{02}^{\beta_1}\psi_{02}^{\beta_1} + \Phi_{02}^{\beta_2}\psi_{02}^{\beta_2} >_H + o(x_{111}^2 + y_{111}^2) \\
= \dfrac{-C_1\pi^2}{2\alpha}\Phi_{02}^{\beta_1} - \dfrac{C_2\pi^2}{2\alpha}\Phi_{02}^{\beta_2} + o(x_{111}^2 + y_{111}^2) \\
= -\dfrac{\pi^3}{4\alpha}(\dfrac{A_1C_1}{\beta_1} + \dfrac{A_2C_2}{\beta_2})(x_{111}^2 + y_{111}^2) + o(x_{111}^2 + y_{111}^2) \\
= \dfrac{1}{8\sqrt{2}}(A_1C_1 + A_2C_2\tau^{-1})(x_{111}^2 + y_{111}^2) + o(x_{111}^2 + y_{111}^2), \\
< G(\psi_{11}^{\beta_{111},1}, \Psi_{11}^{\beta_{111},2}), \Phi >_H = o(x_{111}^2 + y_{111}^2), \\
< G(\Phi, \psi_{11}^{\beta_{111},1}), \Psi_{11}^{\beta_{111},2} >_H = o(x_{111}^2 + y_{111}^2), \\
< \psi_{11}^{\beta_{111},1}, \Psi_{11}^{\beta_{111},1} >_H = \dfrac{1}{\sqrt{2}}(3 + A_1C_1 + A_2C_2),
\end{cases}
$$

Plugging (6.10) and (6.11) into (6.3) and (6.4) respectively and applying Theorem 4.3, we get the reduced bifurcation equations:

$$
(6.12)\qquad \begin{aligned}
\frac{dx_{111}}{dt} =& \beta_{111}(\lambda)x_{111} - \frac{1}{8}\delta(\lambda,\eta)(x_{111}^3 + x_{111}y_{111}^2) \\
&+ o(x_{111}^3 + y_{111}^3) + O(\beta_{111}(\lambda)(x_{111}^3 + y_{111}^3)),
\end{aligned}
$$

$$
(6.13)\qquad \begin{aligned}
\frac{dy_{111}}{dt} =& \beta_{111}(\lambda)y_{111} - \frac{1}{8}\delta(\lambda,\eta)(x_{111}^2 y_{111} + y_{111}^3) \\
&+ o(x_{111}^3 + y_{111}^3) + O(\beta_{111}(\lambda)(x_{111}^3 + y_{111}^3)),
\end{aligned}
$$

where

$$
(6.14)\qquad \delta(\lambda,\eta) = \frac{(A_1C_1 + A_2C_2\tau^{-1})}{(3 + A_1C_1 + A_2C_2)}.
$$

The following lemma determines the sign of $\delta(\lambda, \eta)$.

Lemma 6.1. (1) *Under the assumption (5.8),*

$$\delta(\lambda,\eta)\begin{cases} >0 & \text{if } \eta<\eta_{c_1},\\ <0 & \text{if } \eta>\eta_{c_1}.\end{cases}$$

(2) *If we replace $\tau>1$ ($\tau\neq\sigma$) in (5.8), then $\delta(\lambda,\eta)>0$ for all $\eta>0$.*

Proof. Step 1. Under the assumption (5.8), $\lambda\approx\lambda_c=\frac{\eta}{\tau}+\frac{27\pi^4}{4}$ and $\beta_{111}\approx 0$. It suffices to prove the lemma for $\lambda=\frac{\eta}{\tau}+\frac{27\pi^4}{4}$ and $\beta_{111}=0$. Note that

$$A_1C_1=\frac{\sigma\lambda}{(\beta_{111}+\gamma_{11}^2)^2}\text{ and }A_2C_2=\frac{-\sigma\eta}{(\beta_{111}+\tau\gamma_{11}^2)^2}.$$

Plugging $\lambda=\frac{\eta}{\tau}+\frac{27\pi^4}{4}$ and $\beta_{111}=0$ into the denominator and numerator of $\delta(\lambda,\eta)$ respectively yileds

$$\begin{aligned}3+A_1C_1+A_2C_2&\approx 3+\gamma_{11}^{-4}(\sigma\frac{\eta}{\tau}-\sigma\frac{\eta}{\tau^2})\\&=\sigma\gamma_{11}^{-4}(1-\tau)\tau^{-2}(\eta_c-\eta)\\&>0,\\ \text{and}\qquad A_1C_1+A_2C_2\tau^{-1}&\approx\sigma\gamma_{11}^{-4}(1-\tau^2)\tau^{-3}(\eta_{c_1}-\eta).\end{aligned}$$

Hence the sign of $\delta(\lambda,\eta)$ is determined by $(\eta_{c_1}-\eta)$.

Step 2. If $\tau>1$, we have

$$\begin{aligned}3+A_1C_1+A_2C_2&\approx 3+\gamma_{11}^{-4}(\sigma\frac{\eta}{\tau}-\sigma\frac{\eta}{\tau^2})>0,\\ \text{and}\qquad A_1C_1+A_2C_2\tau^{-1}&\approx\sigma\gamma_{11}^{-4}(\frac{27}{4}\pi^4-\eta(1-\tau^2)\tau^{-3})>0.\end{aligned}$$

Hence, $\delta(\lambda,\eta)>0$ for all $\eta>0$. This completes the proof. □

7. Completion of the proofs

We demonstrate the proof of Theorem 3.3; Theorem 3.5 can be proved in the same fashion.

7.1. **S^1-Attractor.** First, Assertion (1) of Theorem 3.3 follows from (6.12)-(6.14) and Lemma 6.1. Then, by Theorem 4.2, the equations bifurcate from $(0,\lambda_c)$ to an attractor Σ_λ for $\lambda>\lambda_c$, which is homeomorphic to S^1.

7.2. **Singularity Cycle.** We shall prove that the bifurcated attractor Σ_λ of (2.4)-(2.8) given in Theorem 3.3 is a cycle of steady state solutions. First, let

$$\begin{aligned}&H'=\{(U,T,S)|u(-x,z)=-u(x,z)\},\\&H_1'=H_1\cap H'.\end{aligned}$$

It is well-known that H' and H_1' are invariant spaces for the operator $L_{\lambda\eta}+G$ given by (2.10) in the sense that

$$L_{\lambda\eta}+G:H_1'\to H'.$$

It is clear that the first eigenvalue of $L_{\lambda\eta}|H_1'$ is simple when $\eta<\eta_c$. By the Kransnoselski bifurcation theorem (see aomng others Chow and Hale [2] and Nirenberg [10]), when λ crosses λ_c, the equations bifurcate from the trivial solution to a steady state solution in H'. Therefore the attractor Σ_λ contains at least one

steady state solution. Secondly, it's easy to check that the equations (2.4)-(2.8) are translation invariant in x-direction. Hence if $\psi_0(x,z) = (U(x,z),T(x,z),S(x,z))$ is a steady state solution, then $\psi_0(x+\rho,z)$ are steady state solutions as well. By periodic conditions in x-direction, the set

$$S_{\psi_0} = \{\psi_0(x+\rho,z) | \rho \in \mathbb{R}\}$$

is a cycle homeophic to S^1 in H_1. Therefore the steady state of (2.4)-(2.8) generates a cycle of steady state solutions. Hence the bifurcated attractor Σ_λ contains at least a cycle of steady state solutions.

7.3. **Asymptotic structure of solutions.** It's easy to see that for any initial value $\psi_0 = (U_0, T_0, S_0) \in H$, there is a time $t_1 \geq 0$ such that the solution $\psi = (U(t,\psi_0), T(t,\psi_0), S(t,\psi_0))$ is C^∞ for $t > t_1$, and is uniformly bounded in C^r-norm for any given $r \geq 1$. Hence, by Theorem 4.1, we have

$$\lim_{t\to\infty} \min_{\phi\in\Sigma_\lambda} \|\psi(t,\psi_0) - \phi\|_{C^r} = 0. \tag{7.1}$$

We infer then from (6.12) and (6.14) that for any steady state solution $\phi = (e,T,S) \in \Sigma_\lambda$ of (2.4)-(2.8), the vector field $e = (e_1, e_2)$ can be expressed as

$$\begin{cases} e_1 = -r\sqrt{2}\sin(\alpha x+\theta)\cos\pi z + v_1(x_{111}, y_{111}, \beta_{111}), \\ e_2 = r\cos(\alpha x+\theta)\sin\pi z + v_2(x_{111}, y_{111}, \beta_{111}), \end{cases} \tag{7.2}$$

for some $0 \leq \theta \leq 2\pi$. Here

$$\begin{cases} r = \sqrt{x_{111}^2 + y_{111}^2} = \sqrt{\dfrac{8\beta_{111}(\lambda)}{\delta}} + o(\sqrt{\beta_{111}(\lambda)}) & \text{if } \lambda > \lambda_c, \\ v_i(x_{111}, y_{111}, \beta_{111}) = o(\sqrt{\beta_{111}(\lambda)}) & \text{for } i = 1,2. \end{cases} \tag{7.3}$$

Now we show that the vector field

$$e_0 = \left(-r\sqrt{2}\sin\alpha x\cos\pi z, r\cos\alpha x\sin\pi z\right) \tag{7.4}$$

is regular in $\Omega = \mathbb{R}^1 \times (0,1)$. The singular points of e are $(x,z) = ((k+\frac{1}{2})\sqrt{2}, \frac{1}{2})$, $(k\sqrt{2},1)$ and $(k\sqrt{2},0)$ with $k \in \mathbb{Z}$, and

$$\begin{aligned} detDe_0(x,z) &= det\begin{pmatrix} -\sqrt{2}r\alpha\cos\alpha x\cos\pi z & \sqrt{2}r\pi\sin\alpha x\sin\pi z \\ -r\alpha\sin\alpha x\sin\pi z & r\pi\cos\alpha x\cos\pi z \end{pmatrix} \\ &= \begin{cases} r^2\pi^2 \neq 0, \text{ for } (x,z) = ((k+\dfrac{1}{2})\sqrt{2}, \dfrac{1}{2}), \\ -r^2\pi^2 \neq 0, \text{ for } (x,z) = (k\sqrt{2},0),\ (k\sqrt{2},1). \end{cases} \end{aligned}$$

Therefore the vector field (7.4) is regular, and consequently, the vector field e in (7.2) is regular for any $\lambda_c < \lambda < \lambda_c + \epsilon$ for some ϵ small. It follows from Theorem 4.7 that the vector field e of (7.2) is topologically equivalent to the vector field e_0 given by (7.4), which has the topological structure as shown in Figure 3.5.

7.4. **Proof of Theorem 3.5.** Inferring from (6.12)-(6.14) and Lemma 6.1, the equations bifurcate from $(0,\lambda_c)$ to a repeller Σ_λ^1 for $\lambda < \lambda_c$, which is homeomorphic to S^1. Since $(U,T,S) = (0,0,0)$ is a global attractor for λ near 0 , there exists a saddle-node bifurcation point λ_0 in between 0 and λ_c. This completes the proof.

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